ETMAG

CORONALECTURE 8

Subspaces

Linear independence

Bases, dimension

May 11, 12:15

Examples. (of subspaces or not-subspaces)
Decide which subsets are subspaces:

1. $\{(x, y) \in \mathbb{R}^2 : xy \geq 0\}$ in $\mathbb{R}^2$ over $\mathbb{R}$
2. $\{(x, y) \in \mathbb{R}^2 : x + y \geq 0\}$ in $\mathbb{R}^2$ over $\mathbb{R}$
3. $\{(x, y) \in \mathbb{R}^2 : x = 5y\}$ in $\mathbb{R}^2$ over $\mathbb{R}$
4. $\{(x, y) \in \mathbb{R}^2 : x^2 = y\}$ in $\mathbb{R}^2$ over $\mathbb{R}$
5. $\{(x, y, z) \in \mathbb{R}^3 : x + y - 3z = 1\}$ in $\mathbb{R}^3$ over $\mathbb{R}$
6. $\left\{\{a, b\}, \{a\}, \varnothing\right\}$ in $2^{\{a,b,c\}}$ over $\mathbb{Z}_2$

Definition

Let V be a vector space over a field K and let $a_1, \dots, a_n \in$K and $v_1, \dots, v_n \in$ V. The vector $a_1 v_1 + \cdots + a_n v_n$ is called the linear combination of vectors $v_1, \dots, v_n$ with coefficients $a_1, \dots, a_n$.

A common problem in linear algebra is to decide whether a vector is or is not a linear combination of other vectors.

Example.

You decide to blow-up Polytechnica. You find a recipe on the dark net which says you need a mixture of 30% of ingredient A, 50% of B and 20% of C. A leading branch of toothpaste T consists of 10, 60 and 30 percent of those, a scouring powder S has 5, 80 and 15 and a washing machine powder P has 25, 50 and 25. Can you mix your explosive using those?

In the language of vectors we are asking if there exist coefficients t, s ,p such that

$$(30,50,20) = t(10,60,30) + s(5,80,15) + p(25,50,25)$$

i.e. (30,50,20) is a linear combination of (10,60,30), (5,80,15) and (25,50,25).

In this example we must also require the coefficients to be ≥ 0.

Comparing the left to the right hand side of

$(30,50,20) = t(10,60,30) + s(5,80,15) + p(25,50,25)$

component by component we clearly get a system of equations

$$\begin{cases} 30 = 10t + 5s + 25p \\ 50 = 60t + 80s + 50p \\ 20 = 30t + 15s + 25p \end{cases}$$

We can word it slightly differently: does our vector (30,50,20) belong to the set of all possible linear combinations of (10,60,30), (5,80,15) and (25,50,25)

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Before we switch to the other presentation – no, we cannot blow-up Polytechnica. We cannot get 30% of A in our mixture because in each of the household chemicals the content of A is below 30%.

Now we switch to slide #16 of the other presentation.

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We switch to the old presentation for linear independence of sets of vectors, bases and dimension.

Definition. (alternate definition od *span*)
Let $S \subseteq V$ (a sub**set**, not necessarily a sub**space**). Then by *span*(*S*) we denote the smallest subspace of V containing S.

We call span(S) the *subspace spanned by S.*

One advantage of this definition over the other one is it covers the case $S = \varnothing$ without branching.

Fact.

Let V(S) denote the set of all subspaces of V containing S. Then

$$span(S) = \bigcap_{T \in V(S)} T$$

Proof. It is enough to show that intersection of a collection of subspaces is a subspace of V and that is easy. (All contain Θ so intersection does, too, etc.)

Theorem.

The set $S=\{v_1,v_2, \dots ,v_n\}$ is linearly independent iff no vector from S is a linear combination of the others.

<u>**Proof.**</u> ($\Rightarrow$) Suppose one of the vectors is a linear combination of the others. Without loss of generality we may assume that v_n is the one, i.e. $v_n=a_1v_1+a_2v_2+ \dots .a_{n-1}v_{n-1}$. Then we may write $\Theta=a_1v_1+a_2v_2+ \dots .a_{n-1}v_{n-1}+(-1)v_n$. Since $-1 \neq 0$ the set $\{v_1,v_2, \dots ,v_n\}$ is linearly dependent.

($\Leftarrow$) Suppose now that $\{v_1,v_2, \dots ,v_n\}$ is linearly dependent, i.e. there exist coefficients $a_1,a_2, \dots ,a_n$, not all of them zeroes, such that $\Theta=a_1v_1+a_2v_2+ \dots .a_nv_n$. Again, without losing generality, we may assume that $a_n\neq0$ (we can always renumber the vectors so that the one with nonzero coefficient is the last). Since nonzero scalars are invertible, we have $v_n=(-a_1a_n^{-1})v_1 + (-a_2a_n^{-1})v_2 + \dots +(-a_{n-1}a_n^{-1})v_{n-1}$